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Integral means induced by interpolation paths of operator means

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Throughout this paper, capital letters stand for $n\times n$ (complex) positive-definite matrices. For positive operator monotone function f_{m} with $f_{\mathrm{m}}(1)=1$ on $(0,\infty)$, which is called *the representing function for* m, the *Kubo-Ando* (*operator*) *mean* [15] is defined by

$$A\,\mathrm{m}\,B=A^{\frac{1}{2}}f_{\mathrm{m}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

The quasi-arithmetic Kubo-Ando mean is

$$A\#_{r,t}B=A^{\frac{1}{2}}\left((1-t)I+t\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{r}\right)^{\frac{1}{r}}A^{\frac{1}{2}}$$

Another type of quasi-arithmetic operator mean is

$$A\diamondsuit_{r,t}B=((1-t)A^{r}+tB^{r})^{\frac{1}{r}}.$$

The latter paths are the geodesic of the Hiai-Petz geometry [12] and one of the former paths for $r=0$;

$$A\#_{t}B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{t}A^{\frac{1}{2}}$$

is the geodesic of the CPR geometry [1, 2]. Here we consider such types of operator means here and call the latter type *chaotical means* [9, 3].

Considering Uhlmann's entropy, these paths are *interpolational* ([7])

$$(A\,\mathrm{m}_{r}B)\,\mathrm{m}_{t}(A\,\mathrm{m}_{s}B)=A\,\mathrm{m}_{(1-t)r+ts}B. \tag{1}$$

Interpolational paths $A\,\mathrm{m}_tB$ are (operator-valued) convex for t and differentiable ([8]) and determine the relative (operator) entropy as the derivatives at the end points ([7, 3, 17]). Thus interpolational paths play important parts in geometric view for matrices or operators. So we discuss properties around interpolational paths in this paper.

For a *symmetric* mean m (i.e., $A\,\mathrm{m}\,B=B\,\mathrm{m}\,A$), we can define the (continuous) path from A to B by the following binary construction: Based on a initial condition

$$A\,\mathrm{m}_0B=A,\quad A\,\mathrm{m}_{1/2}B=A\,\mathrm{m}\,B,\quad A\,\mathrm{m}_1B=B,$$

define operator means for binary fractions for $2k+1<2^{n+1}$ inductively:

$$A\,\mathrm{m}_{(2k+1)/2^{n+1}}B=(A\,\mathrm{m}_{k/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{(k+1)/2^n}B)=(A\,\mathrm{m}_{(k+1)/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{k/2^n}B). \tag{2}$$

Chaotical operator means are interpolational, but Kubo-Ando means are not: The logarithmic Kubo-Ando mean

$$A\,\mathbf{L}\,B = A^{\frac{1}{2}}\ell\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

where $\ell(x) = (x-1)/\log x$ is not interpolational. One of the equivalent condition that a mean is interpolational is ([5]):

Theorem 1. *A symmetric mean* m *is interpolational if and only if*

mixing property: $\qquad (a\,\mathrm{m}\,b)\,\mathrm{m}\,(c\,\mathrm{m}\,d) = (a\,\mathrm{m}\,c)\,\mathrm{m}\,(b\,\mathrm{m}\,d)$

holds for all positive numbers a, b, c *and* d.

Proof. Suppose m_t is an interpolational path. By the homogeneity, we may assume that $d = 1$, $a > b, c > 1$. Then there exist $r, s > 0$ with $b = 1\,\mathrm{m}_r a$ and $c = 1\,\mathrm{m}_s a$. It follows that

$$\begin{aligned}(a\,\mathrm{m}\,b)\,\mathrm{m}\,(c\,\mathrm{m}\,1) &= (a\,\mathrm{m}\,(1\,\mathrm{m}_r a))\,\mathrm{m}\,((1\,\mathrm{m}_s a)\,\mathrm{m}\,1)\\ &= (1\,\mathrm{m}_{(r+1)/2}a)\,\mathrm{m}\,(1\,\mathrm{m}_{s/2}a) = 1\,\mathrm{m}_{(r+s+1)/4}a\\ &= (1\,\mathrm{m}_{(s+1)/2}a)\,\mathrm{m}\,(1\,\mathrm{m}_{r/2}a)\\ &= (a\,\mathrm{m}\,(1\,\mathrm{m}_s a))\,\mathrm{m}\,((1\,\mathrm{m}_r a)\,\mathrm{m}\,1) = (a\,\mathrm{m}\,c)\,\mathrm{m}\,(b\,\mathrm{m}\,1).\end{aligned}$$

Conversely suppose m satisfies the mixing property. First we show

$$(1\,\mathrm{m}_{k/2^n}a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n}a) = 1\,\mathrm{m}_{(k+\ell)/2^{n+1}}a \tag{3}$$

inductively. It holds for $n = 1$. Suppose it holds for not greater than n. We may assume that the k and ℓ are odd numbers $2k+1$ and $2\ell+1$ respectively. Then, by the definition (2), the mixing property and symmetry, we get

$$\begin{aligned}&(1\,\mathrm{m}_{(2k+1)/2^{n+1}}a)\,\mathrm{m}\,(1\,\mathrm{m}_{(2\ell+1)/2^{n+1}}a)\\ &= \big((1\,\mathrm{m}_{k/2^n}a)\,\mathrm{m}\,(1\,\mathrm{m}_{(k+1)/2^n}a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_{(\ell+1)/2^n}a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n}a)\big)\\ &= \big((1\,\mathrm{m}_{k/2^n}a)\,\mathrm{m}\,(1\,\mathrm{m}_{(\ell+1)/2^n}a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_{(k+1)/2^n}a)\,\mathrm{m}\,(1\,\mathrm{m}_{\ell/2^n}a)\big)\\ &= (1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}}a)\,\mathrm{m}\,(1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}}a) = 1\,\mathrm{m}_{(k+\ell+1)/2^{n+1}}a,\end{aligned}$$

so that (3) holds for all n. By the continuity, we have

$$(1\,\mathrm{m}_r a)\,\mathrm{m}\,(1\,\mathrm{m}_s a) = 1\,\mathrm{m}_{(r+s)/2}a \tag{3'}$$

for all $r, s \in [0,1]$. Similarly we show

$$(1\,\mathrm{m}_r a)\,\mathrm{m}_{k/2^n}(1\,\mathrm{m}_s a) = 1\,\mathrm{m}_{(1-k/2^n)r+(k/2^n)s}a \tag{4}$$

inductively. In fact,

$$\begin{aligned}
&(1\,\mathrm{m}_r a)\,\mathrm{m}_{(2k+1)/2^{n+1}}(1\,\mathrm{m}_s a)\\
&\quad = \big((1\,\mathrm{m}_r a)\,\mathrm{m}_{k/2^n}(1\,\mathrm{m}_s a)\big)\ \mathrm{m}\ \big((1\,\mathrm{m}_r a)\,\mathrm{m}_{(k+1)/2^n}(1\,\mathrm{m}_s a)\big)\\
&\quad = \left((1\,\mathrm{m}_{\left((2^n-k)r+ks\right)/2^n} a)\right)\ \mathrm{m}\ \left((1\,\mathrm{m}_{\left((2^n-(k+1))r+(k+1)s\right)/2^n} a)\right)\\
&\quad = 1\,\mathrm{m}_{\left((2^{n+1}-(2k+1))r+(2k+1)s\right)/2^{n+1}} a = 1\,\mathrm{m}_{\left(1-(2k+1)/2^{n+1}\right)r+\left((2k+1)/2^{n+1}\right)s} a,
\end{aligned}$$

so that (4) holds, and hence we have the interpolationality by the continuity. $\square$

Putting $b = x$, $c = y$ and $a = d = 1$, we immediately obtain,

Corollary 2. *If* m *is interpolational, then*

(5) $$f(x\,\mathrm{m}\,y) = f(x)\,\mathrm{m}\,f(y), \quad \textit{that is,} \quad x\,\mathrm{m}\,y = f^{-1}\big(f(x)\,\mathrm{m}\,f(y)\big).$$

Every chaotical operator mean satisfies

operator mixing property: $(A\,\mathfrak{m}\,B)\,\mathfrak{m}\,(C\,\mathfrak{m}\,D) = (A\,\mathfrak{m}\,C)\,\mathfrak{m}\,(B\,\mathfrak{m}\,D)$

and

(5′) $$f(A\,\mathfrak{m}\,B) = f(A)\,\mathfrak{m}\,f(B), \quad \text{that is,} \quad A\,\mathfrak{m}\,B = f^{-1}\big(f(A)\,\mathfrak{m}\,f(B)\big).$$

But, if a Kubo-Ando operator mean satisfies the above, then we can show that it is an arithmetic or a harmonic one ([5]). Recall that the adjoint m^* is defined by $A\,\mathrm{m}^*B = (A^{-1}\,\mathrm{m}B^{-1})^{-1}$:

Theorem 3. *If a symmetric operator mean* m *satisfies the operator mixing property, then it coincides with the arithmetic mean* ∇ *or the harmonic one* $!$.

Proof. For the representing function f of m, we may assume $f(0) = 0$. Now we will show m is the harmonic mean. For $x > 0$ and projections of rank one

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q = \frac{1}{1+x}\begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix} \quad \text{and} \quad R = \frac{1}{1+f(x)}\begin{pmatrix} f(x) & \sqrt{f(x)} \\ \sqrt{f(x)} & 1 \end{pmatrix},$$

put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \quad \text{and} \quad B = P.$$

Since $\|BA^{-1}B\| = \frac{1}{2}(1 + 1/x) = \frac{1+x}{2x}$ and m is symmetric, we have

$$\begin{aligned}
A\,\mathrm{m}\,B &= f\left(\frac{1+x}{2x}\right)\frac{2x}{1+x}P = f\left(\frac{2x}{1+x}\right)P,\\
f\left(A\,\mathrm{m}\,B\right) &= f\left(f\left(\frac{2x}{1+x}\right)\right)P = f(f(h(x)))P \quad \text{and}\\
f(A)\,\mathrm{m}\,f(B) &= f(A)\,\mathrm{m}\,B = f\left(\frac{2f(x)}{1+f(x)}\right)P = f(h(f(x)))P,
\end{aligned}$$

so that $f(h(x)) = h(f(x))$, or equivalently, $f^*(a(x)) = a(f^*(x))$. Since f^* is concave, then f^* is affine on the interval between 1 and x, and hence $f^*(x) = \alpha + \beta x$ for all $x > 0$. By the symmetric condition, we have

$$\alpha + \beta x = f^*(x) = x f^*(1/x) = \alpha x + \beta$$

for all $x > 0$. Thus $\alpha = \beta = 1/2$ by $f^*(1) = 1$, that is, m^* is the arithmetic mean. Therefore m is the harmonic one. □

For the numerical case, it is uncertain whether (5) implies the mixing property or not. But, by the proof of Theorem 8, we have that (5') implies the operator mixing property since the means ∇ and $!$ have its property.

Corollary 4. *If a symmetric operator mean satisfies* (5′), *then it has the operator mixing property.*

For (interpolational) paths m_t, we can define the *induced integral mean* $\widetilde{\mathrm{m}}$ by

$$A \,\widetilde{\mathrm{m}}\, B = \int_0^1 A \,\mathrm{m}_t B \, dt.$$

Then we obtain ([5]):

Theorem 5. *If* m *is interpolational, then* $\widetilde{\mathrm{m}}$ *is symmetric and not greater than* m *itself.*

Proof. Since $B \,\mathrm{m}_t A = A \,\mathrm{m}_{1-t} B.$, we have

$$A \,\widetilde{\mathrm{m}}\, B = \int_0^1 A \,\mathrm{m}_t B \, dt = \int_0^1 B \,\mathrm{m}_{1-t} A \, dt = \int_0^1 B \,\mathrm{m}_s A \, ds = B \,\widetilde{\mathrm{m}}\, A.$$

By the maximality of the arithmetic mean, we have

$$\begin{aligned} A \,\widetilde{\mathrm{m}}\, B &= \int_0^1 A \,\mathrm{m}_t B \, dt = \frac{\int_0^1 A \,\mathrm{m}_t B \, dt + \int_0^1 B \,\mathrm{m}_t A \, dt}{2} = \int_0^1 \frac{A \,\mathrm{m}_t B + A \,\mathrm{m}_{1-t} B}{2} \, dt \\ &\geq \int_0^1 (A \,\mathrm{m}_t B) \,\mathrm{m}\, (A \,\mathrm{m}_{1-t} B) \, dt = \int_0^1 A \,\mathrm{m}\, B \, dt = A \,\mathrm{m}\, B. \end{aligned}$$

Example. Put operator monotone functions $f_t^{(r)}(x) = (1 - t + t x^r)^{\frac{1}{r}} \quad (-1 \leqq r \leqq 1)$, then the representing functions $\widetilde{f}^{(r)}$ of the induced Kubo-Ando integral means $\widetilde{\#}^{(r)}$ are:

$$\widetilde{f}^{(r)}(x) = \int_0^1 (1 - t + t x^r)^{\frac{1}{r}} \, dt = \left[\frac{(1 - t + t x^r)^{\frac{1+r}{r}}}{(x^r - 1)\frac{1+r}{r}} \right]_0^1 = \frac{r}{1+r} \frac{x^{r+1} - 1}{x^r - 1}.$$

For example,

$$\begin{aligned} &(r = 1) && \text{arithmetic mean:} && \widetilde{f}^{(1)}(x) = \frac{1+x}{2}, \\ &(r = 0) && \text{logarithmic mean:} && \widetilde{f}^{(0)}(x) \equiv \lim_{\varepsilon \downarrow 0} \widetilde{f}^{(\varepsilon)}(x) = \frac{x-1}{\log x}, \\ &(r = -1/2) && \text{geometric mean:} && \widetilde{f}^{(-1/2)}(x) = \sqrt{x}, \\ &(r = -1) && \text{adjoint logarithmic mean:} && \widetilde{f}^{(-1)}(x) \equiv \lim_{\varepsilon \downarrow 0} \widetilde{f}^{(\varepsilon - 1)}(x) = \frac{x \log x}{x - 1}. \end{aligned}$$

It satisfies the following estimation ([6]):

Theorem 6. *For $s \in [0,1]$, $A \,\widetilde{\mathrm{m}}\, B \leq \dfrac{sA + (1-s)B + A \,\mathrm{m}_s B}{2}$.*

In particular, $A \,\widetilde{\mathrm{m}}\, B \leq \dfrac{A \nabla B + A \,\mathrm{m}\, B}{2}$.

Proof. For $\phi(t) = A \,\mathrm{m}_t B$, the convexity of ϕ shows

$$\begin{cases} \frac{t}{s}(\phi(s) - \phi(0)) + \phi(0) & \text{if } 0 \leqq t \leqq s, \text{ and} \\ \frac{t-s}{1-s}(\phi(1) - \phi(s)) + \phi(s) & \text{if } s \leqq t \leqq 1. \end{cases}$$

It follows that

$$\begin{aligned}
\int_0^s A \,\mathrm{m}_t B dt &\leq \int_0^s \left(\frac{t}{s}(A \,\mathrm{m}_s B - A) + A \right) dt = \left[\frac{t^2}{2s}(A \,\mathrm{m}_s B - A) + tA \right]_0^s \\
&= \frac{s^2}{2s}\,(A \,\mathrm{m}_s B - A) + sA = \frac{s}{2}\,(A \,\mathrm{m}_s B + A) \quad \text{and} \\
\int_s^1 A \,\mathrm{m}_t B dt &\leq \int_s^1 \left(\frac{t-s}{1-s}\,(B - A \,\mathrm{m}_s B) + A \,\mathrm{m}_s B \right) dt \\
&= \left[\frac{t^2/2 - ts}{1-s}\,(B - A \,\mathrm{m}_s B) + tA \,\mathrm{m}_s B \right]_s^1 \\
&= \frac{1/2 - s - s^2/2 + s^2}{1-s}\,(B - A \,\mathrm{m}_s B) + (1-s) A \,\mathrm{m}_s B = \frac{1-s}{2}\,(B + A \,\mathrm{m}_s B).
\end{aligned}$$

Therefore, $A \widetilde{\mathrm{m}} B = \displaystyle\int_0^1 A \,\mathrm{m}_t B dt \leq \frac{sA + (1-s)B + A \,\mathrm{m}_s B}{2}$. □

Putting $s = 1/2$, we obtain

Corollary 7.

$$0 \leq A \widetilde{\mathrm{m}} B - A \nabla B \leq \frac{A \nabla B - A \,\mathrm{m}\, B}{2}.$$

Similarly we have

Theorem 8. *For $0 \equiv t_0 < t_1 < \cdots < t_n < t_{n+1} \equiv 1$,*

$$A \,\widetilde{\mathrm{m}}\, B \leq \frac{1}{2} \left(t_1 A + (1 - t_n) B + \sum_{k=1}^{n} (t_{k+1} - t_{k-1}) A \,\mathrm{m}_{t_k} B \right).$$

In particular, $A \,\widetilde{\mathrm{m}}\, B \leq \dfrac{1}{n+1} \left(A \nabla B + \displaystyle\sum_{k=1}^{n} A \,\mathrm{m}_{k/(n+1)} B \right)$.

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